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DETECTING STRUCTURAL PROPERTIES OF FINITE GROUPS BY THE SUM OF ELEMENT ORDERS

BY

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ABSTRACT

In this paper, we introduce a new function related to the sum of element orders of finite groups. It is used to give some criteria for a finite group to be cyclic, abelian, nilpotent, supersolvable and solvable, respectively.

1. Introduction

Given a finite group G, we consider the function

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of x. This has been introduced by H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs [1]. They proved the following theorem:

Theorem A: *If G is a group of order n, then $\psi(G) \leq \psi(C_n)$, and we have equality if and only if G is cyclic.*

In other words, the cyclic group C_n is the unique group of order n which attains the maximal value of $\psi(G)$ among groups of order n.

Since then many authors have studied the function $\psi(G)$ and its relations with the structure of G (see, e.g., [2], [3], [5], [6], [7], [8], [9], [12], [14]). In the papers [4] and [12] M. Amiri and S. M. Jafarian Amiri, and, independently, R. Shen, G. Chen and C. Wu started the investigation of groups with the second

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largest value of the sum of element orders. M. Herzog, P. Longobardi and M. Maj [6] determined the exact upper bound for $\psi(G)$ for non-cyclic groups of order n:

Theorem B: *If G is a non-cyclic group of order n and q is the least prime divisor of the order of n, then*

$$\psi(G) \leq \frac{[(q^2-1)q+1](q+1)}{q^5+1}\,\psi(C_n) = f(q)\psi(C_n).$$

Moreover, the equality holds if and only if $n = q^2m$ with $(m, q!) = 1$ and

$$G \cong (C_q \times C_q) \times C_m.$$

Note that the above function f is strictly decreasing on $[2, \infty)$. Consequently, we have

$$\psi(G) \leq f(2)\psi(C_n) = \frac{7}{11}\,\psi(C_n),$$

and the equality holds for $n = 4m$ with m odd and $G \cong (C_2 \times C_2) \times C_m$.

By using the sum of element orders, several criteria for solvability of finite groups have been determined (see, e.g., [5, 7]). We recall here the following theorem of M. Baniasad Asad and B. Khosravi [5]:

Theorem C: *If G is a group of order n and $\psi(G) > \frac{211}{1617}\,\psi(C_n)$, then G is solvable.*

Note that the equality

$$\psi(G) = \frac{211}{1617}\,\psi(C_n)$$

occurs for $n = 60m$ with $(30, m) = 1$ and $G \cong A_5 \times C_m$.

We also recall a criterion for nilpotency of finite groups that has been proved in [14]:

Theorem D: *If G is a group of order n and $\psi(G) > \frac{13}{21}\,\psi(C_n)$, then G is nilpotent. Moreover, we have $\psi(G) = \frac{13}{21}\,\psi(C_n)$ if and only if $n = 6m$ with $(6, m) = 1$ and $G \cong S_3 \times C_m$.*

The four largest values of the ratio

$$\psi'(G) = \frac{\psi(G)}{\psi(C_{|G|})}$$

and the groups G for which they are attained can be obtained from Theorem D.

Corollary E: *Let G be a finite group satisfying $\psi'(G) > \frac{13}{21}$. Then*

$$\psi'(G) \in \left\{\frac{27}{43}, \frac{7}{11}, 1\right\},$$

and one of the following holds:

(a) *$G \cong Q_8 \times C_m$, where m is odd;*
(b) *$G \cong (C_2 \times C_2) \times C_m$, where m is odd;*
(c) *G is cyclic.*

The above results show that a finite group G becomes cyclic, abelian, nilpotent or solvable if $\psi'(G)$ is sufficiently large[1]. In what follows, we consider the function

$$\psi''(G) = \frac{\psi(G)}{|G|^2}.$$

Clearly, $\psi''(G) < 1$ if G is non-trivial, and there are sequences of groups (G_n) such that $\psi''(G_n)$ tends to 1 when n tends to infinity (for example, (C_p) where p runs over the set of primes). We also observe that ψ'' satisfies the following important property:

$$\psi''(G) \leq \psi''(G/H), \quad \forall\, H \lhd G, \tag{1}$$

by Proposition 2.6 of [7]. We will use this new function to give criteria for a finite group to be cyclic, abelian, nilpotent, supersolvable and solvable, respectively. Our main result is the following theorem.

Theorem 1.1: *Let G be a finite group. Then the following hold:*

(a) *If $\psi''(G) > \frac{7}{16} = \psi''(C_2 \times C_2)$, then G is cyclic.*
(b) *If $\psi''(G) > \frac{27}{64} = \psi''(Q_8)$, then G is abelian.*
(c) *If $\psi''(G) > \frac{13}{36} = \psi''(S_3)$, then G is nilpotent.*
(d) *If $\psi''(G) > \frac{31}{144} = \psi''(A_4)$, then G is supersolvable.*
(e) *If $\psi''(G) > \frac{211}{3600} = \psi''(A_5)$, then G is solvable.*

Note that the converses of the implications in Theorem 1.1 are not true. We observe that $\psi''(C_{p^m})$ tends to $\frac{p}{p+1}$ when m tends to infinity. Let $(p_i)_{i\in\mathbb{N}^*}$ be the sequence of primes. Since $\prod_{i\geq 1} \frac{p_i}{p_i+1} = 0$, there exists a positive integer k such that

$$c = \prod_{i=1}^{k} \frac{p_i}{p_i+1} < \frac{211}{3600}.$$

[1] A similar result for supersolvability has been conjectured in [14].

If

$$n = \prod_{i=1}^{k} p_i^{n_i},$$

then $\psi''(C_n)$ tends to c when $n_1, \ldots, n_k$ tend to infinity. In other words, $\psi''(C_n) < \frac{211}{3600}$ for $n_1, \ldots, n_k$ sufficiently large, i.e., there are cyclic groups G with $\psi''(G) < \frac{211}{3600}$.

For the proof of the above theorem, we need some preliminary results about the function ψ taken from [2, 6].

LEMMA 1.2: *Here G denotes a finite group, p, p_i denote primes and n, n_i, denote positive integers. The following statements hold:*

(1) ([6], Lemma 2.9(1)) $\psi(C_{p^n}) = \frac{p^{2n+1}+1}{p+1}$.
(2) ([6], Lemma 2.2(3)) *ψ is multiplicative, that is, if $G = A \times B$, where A, B are subgroups of G satisfying* $\gcd(|A|, |B|) = 1$, *then* $\psi(G) = \psi(A)\psi(B)$.
(3) ([2], Lemma 2.1) $\psi(A \times B) \leq \psi(A)\psi(B)$. *Moreover,*

$$\psi(A \times B) = \psi(A)\psi(B)$$

if and only if $\gcd(|A|, |B|) = 1$.
(4) ([6], Lemma 2.9(2)) *If $n = \prod_{i=1}^{k} p_i^{n_i}$, where $p_i \neq p_j$ for $i \neq j$, then*

$$\psi(C_n) = \prod_{i=1}^{k} \psi(C_{p_i^{n_i}}).$$

(5) ([6], Lemma 2.2(5)) *If $G = P \rtimes H$, where P is a cyclic p-group, $|H| > 1$ and $(p, |H|) = 1$, then*

$$\psi(G) = |P|\psi(H) + (\psi(P) - |P|)\psi(C_H(P)).$$

We also need the following two theorems: The first one is due to A. Lucchini (see Theorem 2.20 in [10]), while the second one is a consequence of a theorem of B. Huppert and N. Ito (see Theorem 13.10.1 in [11]).

THEOREM 1.3: *Let A be a cyclic proper subgroup of a finite group G, and let $K = \mathrm{Core}_G(A)$. Then*

$$[A : K] < [G : A],$$

and, in particular, if $|A| \geq [G : A]$, then $K > 1$.

Theorem 1.4: *Suppose that a finite group G contains a subgroup B of prime power index and B contains a cyclic subgroup H of index $[B : H] \leq 2$. Then G is solvable.*

We end our paper by indicating a natural open problem concerning the criteria in Theorem 1.1.

Open problem: Determine all finite groups G for which $\psi''(G)$ takes the values $\frac{7}{16}$, $\frac{27}{64}$, $\frac{13}{36}$, $\frac{31}{144}$ and $\frac{211}{3600}$, respectively.

Note that, given $c \in (0, 1) \cap \mathbb{Q}$, the main difficulty in solving this problem is to determine positive integers n such that $\psi''(C_n) = c$.

2. Proofs of the main results

First of all, we give three lemmas that will be useful to us.

Lemma 2.1: *Let G be a finite group. If $\psi''(G) \geq \frac{1}{3}$, then either G is cyclic or there exists $x \in G$ such that $[G : \langle x\rangle] = 2$.*

Proof. From the condition $\psi''(G) \geq \frac{1}{3}$ we infer that there exists $x \in G$ such that $o(x) > \frac{|G|}{3}$. This leads to $[G : \langle x\rangle] < 3$, that is, $[G : \langle x\rangle] \in \{1, 2\}$. Then either $G = \langle x\rangle$ is cyclic or $[G : \langle x\rangle] = 2$, as desired. ∎

Lemma 2.2: *Let G be a finite 2-group having a cyclic maximal subgroup. Then the following hold:*

(a) *If $\psi''(G) > \frac{7}{16}$, then G is cyclic.*
(b) *If $\psi''(G) > \frac{27}{64}$, then G is cyclic or $G \cong C_2 \times C_2$.*
(c) *If $\psi''(G) > \frac{13}{36}$, then G is cyclic or $G \cong C_2 \times C_2$ or $G \cong Q_8$.*

Proof. Assume that G is not cyclic and let $n = |G|$. Then, by Theorem 4.1 of [13], II, we infer that either G is abelian of type $C_2 \times C_{2^{n-1}}$, $n \geq 2$, or non-abelian of one of the following types:

$$\begin{aligned}
M(2^n) =&\langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{2^{n-2}+1}\rangle, \quad n \geq 4;\\
D_{2^n} =&\langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{-1}\rangle, \quad n \geq 3;\\
Q_{2^n} =&\langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1}\rangle, \quad n \geq 3;\\
S_{2^n} =&\langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{2^{n-2}-1}\rangle, \quad n \geq 4.
\end{aligned}$$

If $G \cong C_2 \times C_{2^{n-1}}$, then

$$\psi''(G) = \frac{2^{2n}+5}{3\cdot 2^{2n}} > \frac{13}{36} \Leftrightarrow 2^{2n} < 60 \Leftrightarrow n = 2, \text{ i.e., } G \cong C_2 \times C_2,$$

while if G is non-abelian, then we get:

$$\begin{aligned}
\psi''(M(2^n)) =& \frac{1}{2^n} < \frac{13}{36}, \quad \forall\, n \geq 4;\\
\psi''(D_{2^n}) =& \frac{2^{2n-1}+3\cdot 2^n+1}{3\cdot 2^{2n}} < \frac{13}{36}, \quad \forall\, n \geq 3;\\
\psi''(Q_{2^n}) =& \frac{2^{2n-1}+3\cdot 2^{n+1}+1}{3\cdot 2^{2n}} > \frac{13}{36} \Leftrightarrow n = 3, \text{ i.e., } G \cong Q_8;\\
\psi''(S_{2^n}) =& \frac{2^{2n-1}+9\cdot 2^{n-1}+1}{3\cdot 2^{2n}} < \frac{13}{36}, \quad \forall\, n \geq 4.
\end{aligned}$$

This completes the proof. ∎

Lemma 2.3: *Let G be a non-trivial semidirect product of C_{p^n} by C_2, where p is an odd prime and n is a positive integer. Then $\psi''(G) \leq \frac{13}{36}$, and the equality occurs if and only if $p = 3$ and $n = 1$, i.e., $G \cong S_3$.*

Proof. Lemma 1.2, (5), shows that

$$\psi(G) = \psi(C_{p^n} \rtimes C_2) = p^n\psi(C_2) + (\psi(C_{p^n}) - p^n)\psi(C_{C_2}(C_{p^n})).$$

Since the semidirect product $C_{p^n} \rtimes C_2$ is non-trivial, we have $C_{C_2}(C_{p^n}) = 1$ and so

$$\psi(G) = 3\cdot p^n + \frac{p^{2n+1}+1}{p+1} - p^n = \frac{p^{2n+1}+2\cdot p^{n+1}+2\cdot p^n+1}{p+1},$$

i.e.,

$$\psi''(G) = \frac{p^{2n+1}+2\cdot p^{n+1}+2\cdot p^n+1}{4(p^{2n+1}+p^{2n})}.$$

Now, the inequality $\psi''(G) \leq \frac{13}{36}$ is equivalent to

$$4\cdot p^{2n+1} + 13\cdot p^{2n} - 18\cdot p^{n+1} - 18\cdot p^n - 9 \geq 0.$$

We easily observe that the following function on the real variable,

$$f(x) = 4\cdot x^{2n+1} + 13\cdot x^{2n} - 18\cdot x^{n+1} - 18\cdot x^n - 9,$$

is strictly increasing on $[3,\infty)$. Consequently,

$$f(x) \geq f(3) = 25\cdot 3^{2n} - 72\cdot 3^n - 9 \geq 25\cdot 3^{n+1} - 72\cdot 3^n - 9 = 3^{n+1} - 9 \geq 0$$

and we have equality if and only if $x = 3$ and $n = 1$, completing the proof. ∎

We are now able to prove our main result.

Proof of Theorem 1.1. We first prove item (c). Since $\psi''(G) > \frac{13}{36} \geq \frac{1}{3}$, by Lemma 2.1 it follows that either G is cyclic or there exists $x \in G$ such that $[G : \langle x\rangle] = 2$. Let

$$n = |G| = 2^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where $p_2 < p_3 < \cdots < p_k$ are odd primes. Then, for each $i = 2, \ldots, k$, $\langle x\rangle$ contains a cyclic Sylow p_i-subgroup P_i of G. We have $\langle x\rangle \leq N_G(P_i)$ and so P_i is normal in G. Therefore G has a cyclic normal 2-complement, that is,

$$G = C_m \rtimes H,$$

where $m = \frac{n}{2^{n_1}}$ is odd and H is a Sylow 2-subgroup of G. Note that $\langle x\rangle$ also contains a cyclic normal subgroup M of order 2^{n_1-1}, that is. H possesses a cyclic maximal subgroup. Assume that G is not nilpotent. We will show that $\psi''(G) \leq \frac{13}{36}$, contradicting our hypothesis. We infer that there exists $i \in \{2, \ldots, k\}$ such that the semidirect product $C_{p_i^{n_i}} \rtimes H$ is non-trivial, and by property (1) we may assume that

$$G = C_{p_i^{n_i}} \rtimes H.$$

Then G/M is a non-trivial semidirect product of $C_{p_i^{n_i}}$ by C_2, and (1) and Lemma 2.3 imply that

$$\psi''(G) \leq \psi''\Big(\frac{G}{M}\Big) = \psi''(C_{p_i^{n_i}} \rtimes C_2) \leq \frac{13}{36},$$

as desired. Consequently, G is nilpotent.

Next we prove items (a) and (b). If $\psi''(G) > \frac{7}{16}$, then G is nilpotent by (c). Under the above notations, we have $G = C_m \times H$ and

$$\psi''(H) = \psi''\Big(\frac{G}{C_m}\Big) \geq \psi''(G) > \frac{7}{16}$$

implies that H is cyclic by Lemma 2.2. Thus G is cyclic. Similarly, if

$$\psi''(G) > \frac{27}{64},$$

then one obtains $G = C_m \times H$, where either H is cyclic or $H \cong C_2 \times C_2$. Thus G is abelian.

We prove now item (d). We proceed by induction on $|G|$. Since cyclic-by-supersolvable groups are supersolvable, it suffices to show that G contains a non-trivial cyclic normal subgroup C. Indeed, in this case we would have

$$\psi''\Big(\frac{G}{C}\Big) \geq \psi''(G) > \frac{31}{144}$$

and so G/C would be supersolvable by the inductive hypothesis. It is clear that the condition $\psi''(G) > \frac{31}{144}$ implies that there exists $x \in G$ such that

$$[G : \langle x \rangle] < \frac{144}{31}, \text{ i.e., } [G : \langle x \rangle] \in \{1, 2, 3, 4\}.$$

Obviously, we can choose $C = \langle x \rangle$ for $[G : \langle x \rangle] \in \{1, 2\}$. Assume that $[G : \langle x \rangle] = 3$. If $\mathrm{Core}_G(\langle x \rangle) \neq 1$, then we can choose $C = \mathrm{Core}_G(\langle x \rangle)$, while if $\mathrm{Core}_G(\langle x \rangle) = 1$ we infer that G is supersolvable because it can be embedded in S_3. Assume now that $[G : \langle x \rangle] = 4$. Again, if $\mathrm{Core}_G(\langle x \rangle) \neq 1$, then we can choose $C = \mathrm{Core}_G(\langle x \rangle)$, while if $\mathrm{Core}_G(\langle x \rangle) = 1$ we infer that G can be embedded in S_4. Since A_4 and S_4 are the unique non-supersolvable subgroups of S_4 and

$$\psi''(S_4) = \frac{67}{576} < \psi''(A_4) = \frac{31}{144} < \psi''(G),$$

it follows that G is supersolvable.

Finally, we prove item (e). Similarly to (d), it suffices to show that G contains a non-trivial cyclic normal subgroup. The condition $\psi''(G) > \frac{211}{3600}$ implies that there exists $x \in G$ such that

$$[G : \langle x \rangle] \leq 17.$$

If $[G : \langle x \rangle] \in \{16, 17\}$, then the conclusion follows by Theorem 1.4. So, we can suppose that $[G : \langle x \rangle] \leq 15$. If $|G| \geq 225$, then

$$|\langle x \rangle| \geq \frac{225}{[G : \langle x \rangle]} \geq [G : \langle x \rangle]$$

and therefore $\mathrm{Core}_G(\langle x \rangle)$ is a non-trivial cyclic normal subgroup of G by Theorem 1.3. Suppose now that $|G| < 225$ and that G is non-solvable. Then one of the following holds:

(i) $|G| = 60$ and $G \cong A_5$.

(ii) $|G| = 120$ and $G \cong A_5 \times C_2$ or $G \cong S_5$ or $G \cong \mathrm{SL}(2, 5)$.

(iii) $|G| = 168$ and $G \cong \mathrm{PSL}(2, 7)$.

(iv) $|G| = 180$ and $G \cong A_5 \times C_3$.

If $G \cong A_5 \times C_n$ with $n = 2, 3$, then Lemma 1.2, (3), leads to

$$\psi''(G) \leq \psi''(A_5)\psi''(C_n) < \psi''(A_5),$$

in contradiction to (1). Using GAP, in the other cases we get

$$\psi''(S_5) = \frac{471}{14400} < \frac{211}{3600},$$
$$\psi''(\mathrm{SL}(2,5)) = \frac{663}{14400} < \frac{211}{3600},$$
$$\psi''(\mathrm{PSL}(2,7)) = \frac{715}{28224} < \frac{211}{3600},$$

contradicting again the hypothesis.

The proof of Theorem 1.1 is now complete. ∎

References

[1] H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs, *Sums of element orders in finite groups*, Communications in Algebra **37** (2009), 2978–2980.

[2] H. Amiri and S. M. Jafarian Amiri, *Sums of element orders on finite groups of the same order*, Journal of Algebra and its Applications **10** (2011), 187–190.

[3] S. M. Jafarian Amiri, *Second maximum sum of element orders on finite nilpotent groups*, Communications in Algebra **41** (2013), 2055–2059.

[4] S. M. Jafarian Amiri and M. Amiri, *Second maximum sum of element orders on finite groups*, Journal of Pure and Applied Algebra **218** (2014), 531–539.

[5] M. Baniasad Asad and B. Khosravi, *A criterion for solvability of a finite group by the sum of element orders*, Journal of Algebra **516** (2018), 115–124.

[6] M. Herzog, P. Longobardi and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*, Journal of Pure and Applied Algebra **222** (2018), 1628–1642.

[7] M. Herzog, P. Longobardi and M. Maj, *Two new criteria for solvability of finite groups in finite groups*, Journal of Algebra **511** (2018), 215–226.

[8] M. Herzog, P. Longobardi and M. Maj, *Sums of element orders in groups of order $2m$ with m odd*, Communications in Algebra **47** (2019), 2035–2048.

[9] M. Herzog, P. Longobardi and M. Maj, *The second maximal groups with respect to the sum of element orders*, `https://arxiv.org/abs/1901.09662`.

[10] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in mathematics, Vol. 92, American Mathematical Society, Providence, RI, 2008.

[11] W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[12] R. Shen, G. Chen and C. Wu, *On groups with the second largest value of the sum of element orders*, Communications in Algebra **43** (2015), 2618–2631.

[13] M. Suzuki, *Group Theory, Vols. I, II*, Grundlehren der Mathematischen Wissenschaften, Vols. 247, 247, Springer, Berlin, 1982, 1986.

[14] M. Tărnăuceanu *A criterion for nilpotency of a finite group by the sum of element orders*, `https://arxiv.org/abs/1903.09744`.